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Longitude dependence of the geopotential, deduced from synchronous satellites

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Observations of Syncom 2 and Syncom 3 during seven separate periods of free drift have been used in an attempt to find the effective potential at synchronous height. Although the accelerations are well determined near the two longitudes, 180° E and 300° E, where the observations are clustered, the poor distribution in longitude does not permit a satisfactory determination of individual coefficients. It would be particularly valuable to have observations in the region 0 to 40° E or around either of the two stable points near 70° E and 250° E.

If there is or has been any significant population of dust particles in distant geocentric orbits, it is likely that a proportion will have been captured in the synchronous resonance, and will have accumulated near the stable positions. It is therefore suggested that it would be worth attempting to observe whether there are clouds of dust particles near the stable longitudes, and in the stable plane for synchronous height.

1. Introduction

The slight asymmetry of the Earth about its axis produces significant drift accelerations on nearly synchronous satellites since they are in resonance with all the even tesseral harmonics in the geopotential. Because of the great radial distance the harmonic of lowest l-value, namely the term involving $J_{2,2}$ which corresponds to the ellipticity of the equator, must be dominant. Observations of Syncom 2 have therefore been used (Allan & Piggott 1965; Wagner 1965) in an attempt to get a good determination of $J_{2,2}$ and $\phi_{2,2}$. In the present paper all the observations of Syncom 2 and Syncom 3 have been considered up to mid-March 1965, after which unfortunately no further observations are available. Compared with the previous analysis (Allan & Piggott 1965) one drift arc of Syncom 2 has now been rejected, extra observations have been added to the fifth arc, and three further useful arcs have been included, two for Syncom 3 and one for Syncom 2.

2. Equation of motion for a nearly synchronous satellite

The longitude-dependent part of the Earth's gravitational potential at radial distance r, co-latitude θ and longitude ϕ (relative to the Earth's centre of mass) can be written in the form

$$U = (\mu/r) \sum_{l=2}^{\infty} \sum_{m=1}^{l} J_{lm} (R_E/r)^l P_l^m(\cos\theta) \cos m(\phi - \phi_{lm}), \tag{1}$$

where μ is G times the mass of the Earth, R_E is the mean equatorial radius, and J_{lm} and ϕ_{lm} are the constants associated with the (l, m)th tesseral harmonic.

A nearly synchronous satellite in an inclined orbit is not stationary relative to the Earth, but performs a figure-of-eight motion about the equator, and this figure-of-eight will also be asymmetrical if the orbit is appreciably eccentric. If the position of the satellite, however, is given by the usual elliptic elements a, e, I, M, ω and Ω, then the mean longitude $\bar{\phi}$ relative to the Earth, defined by

$$\phi = M + \omega + \Omega - n_0 t, \tag{2}$$

will vary very slowly, provided only that the satellite is nearly synchronous. In (2) $n_0 t$ is the sidereal hour angle. It can then be shown (Allan 1965) that the motion in longitude is governed by the equation

$$\mathrm{d}^2\bar{\phi}/\mathrm{d}t^2 = \sum_{l,m} (A_{lm}+B_{lm}\,\mathrm{d}\bar{\phi}/\mathrm{d}t)J_{lm}\sin m\,(\bar{\phi}-\phi_{lm})+0(e)+\text{luni-solar effects}, \tag{3}$$

where
$$A_{lm} = 3n^2m(R_E/a)^l P_l^m(0)D_{lm}(I), \tag{4}$$

and
$$B_{lm} = -nm(R_E/a)^l P_l^m(0)\left\{2(l+1)D_{lm}(I)+\tan\tfrac{1}{2}I\,\frac{\mathrm{d}D_{lm}(I)}{\mathrm{d}I}\right\}. \tag{5}$$

The B-terms in (3) give only very small corrections for departures from synchronism which might be significant for arcs with a fast drift rate.

Since $P_l^m(0)$ vanishes for $(l-m)$ odd, (3) contains explicit contributions only from the even tesseral harmonics; the odd harmonics are anti-symmetric about the equator so that their effects cancel out to zero order in the eccentricity. In fact the eccentricity never exceeds about 2×10^{-3} for Syncom 2 and Syncom 3, and the terms of order e in (3) have simply been ignored.

The coefficients $D_{lm}(I)$ in (4) and (5) express how a satellite in an inclined orbit samples the longitudinal force due to the (l, m)th harmonic. They arise from the more general inclination functions $F_{lmp}(I)$ for $(l-m)$ even (Kaula 1961; Allan 1965), according to

$$F_{l,m,\frac{1}{2}(l-m)}(I) = P_l^m(0)D_{lm}(I), \tag{6}$$

where the factor $P_l^m(0)$ is included so that the $D_{lm}(I)$ are normalized to unity at zero inclination. Specific forms up to $l = m = 4$ are as follows:

$$D_{2,2} = c^4;\quad D_{3,1} = c^2(1-10s^2+15s^4);\quad D_{3,3} = c^6;$$
$$D_{4,2} = c^4(1-14s^2+28s^4);\quad D_{4,4} = c^8;$$

where $c \equiv \cos\frac{1}{2}I$, and $s \equiv \sin\frac{1}{2}I$.

3. Data analysis

The observational data used here consists of sets of orbital elements for Syncom 2 and Syncom 3 given at roughly weekly intervals, which are computed from the raw observations by N.A.S.A. Goddard Space Flight Center. Each satellite is allowed to drift freely for a few months before its motion is corrected by an impulse or sequence of impulses. Because of these corrections, the mean longitude during each drift period corresponds to a different solution of (3). Observations have been collected for a total of nine drift periods of reasonable length, seven for Syncom 2 and two for Syncom 3, up to mid-March 1965, after which unfortunately no further information is available. Figure 1 shows the mean longitude as a function of time for these nine arcs.

For any arc it is possible to fit, in the sense of least squares, polynomials in the time to the observed values of $\bar{\phi}$. If the arc covers only a small range of longitude, such as the first arc for Syncom 2 which is 3° long, the sum of squares of the residuals is not significantly reduced (~ 0·5 %) on going from a quadratic to a cubic to a quartic. It is thus possible to find a unique force for a short arc, although it may not be so clear for what longitude it applies. This procedure is less meaningful for longer arcs. Thus on going from a quadratic to a quartic, the sum of squares of residuals falls by 10% for the second arc of Syncom 2 (7° long), and by a factor of more than 5 for the fourth arc (43° long). Accordingly we decided to fit the observations directly to the differential equation (3), so as to make maximum use of the information. If the J_{lm} and ϕ_{lm} were known, and also the initial values of ϕ and $\mathrm{d}\bar{\phi}/\mathrm{d}t$ for each drift period, the subsequent behaviour could be determined from (3). Conversely, by a process of iteration which involves integrating (3) numerically with trial values, the values of the constants, including the initial values of $\bar{\phi}$ and $\mathrm{d}\bar{\phi}/\mathrm{d}t$ for each arc, can be determined to give a least-squares fit to the data.

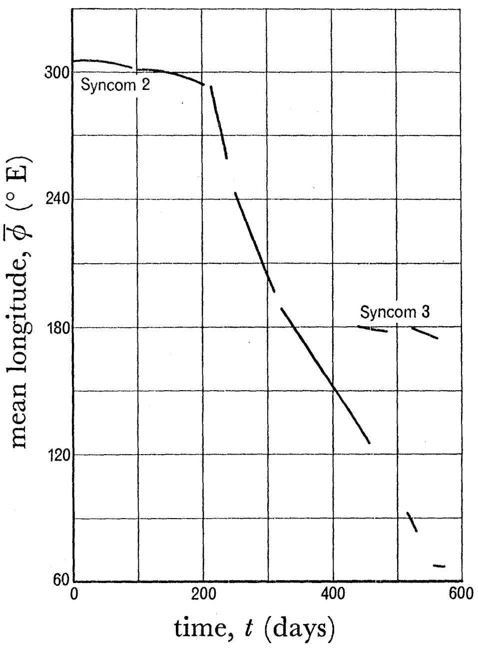

FIGURE 1. Observed mean longitudes of Syncom 2 and Syncom 3.

The coefficients $A_{2,2}$, $A_{3,3}$ and $A_{4,4}$ are all of the order of 10^3 deg/day² for all the arcs, while $A_{3,1}$ and $A_{4,2}$ are very much smaller. Since $J_{4,2}$ would give the same longitude dependence as $J_{2,2}$ it could only be separated by data on satellites at two different inclinations. Thus $J_{4,2}$ has been omitted in the present analysis, since there are only two relatively short arcs for Syncom 3, both at nearly the same longitude. In fact $J_{4,2}$ has negligible effect on Syncom 2, since $D_{4,2}$ vanishes for $I \simeq 34°$.

The small effect of the gravitational attractions of the Sun and Moon on the mean longitude is readily evaluated using the development of Kaula (1962). In the first instance this gives a contribution to $\mathrm{d}^2\bar{\phi}/\mathrm{d}t^2$ to be added to the right of (3), but this is easily integrated analytically to give a luni-solar contribution $\delta\bar{\phi}$ to the change in mean longitude. The largest terms in $\delta\bar{\phi}$ are due to the Sun, with periods of about six months and amplitudes of the order of 0·1°, but the actual values of $\delta\bar{\phi}$ are rather smaller, since there is systematic cancellation. Since the effects are so small, it is enough simply to subtract $\delta\bar{\phi}$ from the observed mean longitude.

In preliminary trials some points consistently gave very high residuals and had to be rejected. Unfortunately the two short arcs 3 and 6 for Syncom 2 had to be removed, as well as six points in the longer arcs. There remained a total of 73 observations in seven arcs.

4. Results

The procedure of least-squares fitting to numerically integrated solutions of (3) has been carried through taking the dominant $J_{2,2}$ term alone, and also together with one and two of the remaining terms $J_{3,1}$, $J_{3,3}$ and $J_{4,4}$ and the results are given in table 1. We have also found the variances and the covariance matrix for the three solutions labelled (*a*), (*b*) and (*c*), and table 2 shows that the values found for individual constants are highly correlated.

Table 1. Solutions for the gravity constants

number of terms	terms	$10^6 J_{lm}$	ϕ_{lm} measured eastwards (deg)	$10^2 \times$ sum of squares of residuals	root mean square residual
1	(2, 2)	1·66	−18·0	2·81	0·020 (*a*)
2	(2, 2) (3, 1)	1·64 8·88	−16·6 −59·5	2·15	0·017 (*b*)
	(2, 2) (3, 3)	1·71 0·07	−17·2 +20·3	2·52	0·019
	(2, 2) (4, 4)	1·73 0·07	−19·2 − 1·9	2·40	0·018
3	(2, 2) (3, 1) (3, 3)	1·64 8·87 0·00	−16·5 −59·5 0·0	2·15	0·017
	(2, 2) (3, 1) (4, 4)	1·63 7·45 0·04	−16·9 −78·7 − 8·5	2·06	0·017
	(2, 2) (3, 3) (4, 4)	1·78 0·07 0·12	−19·3 − 2·9 + 0·2	2·29	0·018 (*c*)

If we take $J_{2,2}$ alone, the observations can be fitted very well with an r.m.s. residual of 0·020° by taking the values

$$10^6 J_{2,2} = 1·66, \quad \phi_{2,2} = -18·0°$$

and the standard deviations are 0·012 and 0·18° respectively. When a second term is included the most significant reduction in the sum of squares of the residuals occurs when the additional term is $J_{3,1}$, as shown by the solution labelled (*b*) in table 1. Since its influence is so slight, it is unlikely that any significance can be attached to the very high value obtained for $J_{3,1}$, although the standard deviation is a factor of four smaller than the value; rather it must reflect the poor distribution of the observations in longitude, most of the points being clustered in two groups near 180° E and 300° E as shown by the histogram in figure 2. The values of $J_{2,2}$ and $\phi_{2,2}$ are little affected by the addition of $J_{3,1}$.

The further reduction in the sum of squares of the residuals on including a third term is not very significant. There is particular interest, however, in the last solution in table 1. This includes the three terms $J_{2,2}$, $J_{3,3}$ and $J_{4,4}$ which one expects to have significant effect on the drift motion, and which one might hope to evaluate from synchronous satellites. It is also to be compared with the solution given previously by Allan & Piggott

TABLE 2. STANDARD DEVIATIONS AND CORRELATION COEFFICIENTS FOR THREE SOLUTIONS

parameter	mean value	standard deviation	correlation coefficient with				
			$\phi_{2,2}$	—	—	—	—
$10^6 J_{2,2}$	1·66	0·012	−0·27	—	—	—	—
$\phi_{2,2}$	−18·0°	0·18°	—	—	—	—	—
			$\phi_{2,2}$	$J_{3,1}$	$\phi_{3,1}$	—	—
$10^6 J_{2,2}$	1·64	0·034	−0·85	−0·30	0·91	—	—
$\phi_{2,2}$	−16·6°	0·66°	—	0·65	−0·82	—	—
$10^6 J_{3,1}$	8·88	2·2	—	—	−0·17	—	—
$\phi_{3,1}$	−59·5°	13°	—	—	—	—	—
			$\phi_{2,2}$	$J_{3,3}$	$\phi_{3,3}$	$J_{4,4}$	$\phi_{4,4}$
$10^6 J_{2,2}$	1·78	0·06	0·28	0·75	0·49	0·26	0·74
$\phi_{2,2}$	−19·3°	1·6°	—	0·50	0·95	−0·78	0·71
$10^6 J_{3,3}$	0·07	0·06	—	—	0·55	0·14	0·59
$\phi_{3,3}$	−2·9°	18°	—	—	—	−0·67	0·82
$10^6 J_{4,4}$	0·12	0·06	—	—	—	—	−0·37
$\phi_{4,4}$	0·2°	5°	—	—	—	—	—

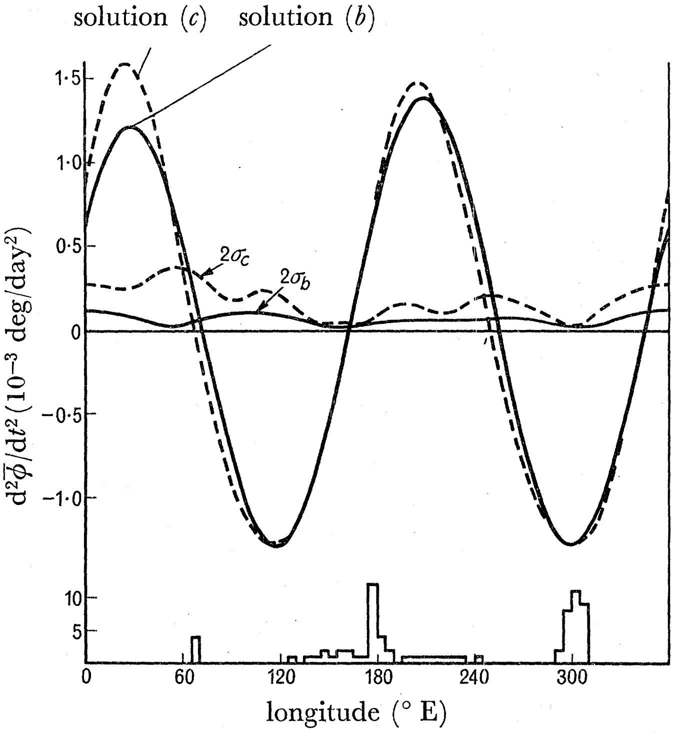

FIGURE 2. Acceleration of a nominal 33° synchronous satellite. (The histogram gives the number of observations per 5° interval.)

(1965) based on the first four arcs for Syncom 2 (one of which, arc 3, is now rejected) and part of the fifth arc. The previous solution gave the values

$$J_{2,2} = 1{\cdot}80 \times 10^{-6} \quad \phi_{2,2} = -15^\circ,$$
$$J_{3,3} = 0{\cdot}18 \times 10^{-6} \quad \phi_{3,3} = 25^\circ,$$
$$J_{4,4} = 0{\cdot}02 \times 10^{-6} \quad \phi_{4,4} = 38^\circ,$$

and $J_{3,3}$ and $J_{4,4}$ are markedly changed in the present solution. Indeed, table 2 shows that the values for $J_{3,3}$ and $J_{4,4}$ are scarcely significant. Wagner (1966) also gives a solution for $J_{2,2}$, $J_{3,1}$ and $J_{3,3}$ in which $J_{2,2}$ and $J_{3,3}$ are in close agreement with our 1965 solution.

The situation regarding the true value of $J_{2,2}$ itself, at least as derived from synchronous satellites, is disappointing. As table 1 shows, the estimated value remains 1·63 to 1·64($\times 10^{-6}$) if $J_{3,1}$ is included but rises to 1·71 to 1·78($\times 10^{-6}$) when $J_{3,1}$ is omitted. It is likely that solution (*c*) is to be preferred, but the firmest conclusion is that more observations are required giving a better distribution in longitude. Thus the attempt to determine $J_{2,2}$ and $\phi_{2,2}$ from synchronous satellites now seems to have been overtaken by results from close orbit satellites. The four most recent determinations by Anderle (1965), Guier & Newton (1965), Kaula (1966) and Gaposchkin (1967) all give values for $J_{2,2}$ and $\phi_{2,2}$ in the region of solution (*c*).

The values we have derived here for the constants are highly correlated. To illustrate this we have computed the acceleration, and the standard deviation thereof, given by the solutions (*b*) and (*c*) as a function of longitude for a nominal 33° synchronous satellite. As figure 2 shows, the two curves agree very well near 180° E and 300° E where the observations are clustered. The standard deviations for both solutions also become very small at these points and the force is clearly determined to better than 1%. The most marked divergence occurs in the region 0 to 40° E, and it would be most valuable to have observations either here or near one or other of the stable positions. It is particularly galling that Syncom 2 is now oscillating freely, with all its fuel expended, around the stable position near 70° E.

Regarding the location of the equilibrium points, the unstable position at 162° E seems to be the most accurately determined (to better than $\pm 1°$), since the solutions (*b*) and (*c*) agree and the standard deviation is very small. For the stable positions, solution (*b*) gives ($72 \pm 1°$) E and ($254 \pm 1°$) E in round figures, and solution (*c*) gives ($67 \pm 3°$) E and ($249 \pm 3°$) E.

There is an interesting possibility concerning the positions of stable equilibrium recently noted by one of us (Allan 1967). It must be emphasized that a satellite oscillating about a stable point is physically captured and is not displaced by small drag forces. The situation is similar to the restricted three-body problem, for the ellipticity of the equator could be crudely pictured as arising from two extra masses at the extremities of the major axis of the Earth's equatorial section. The stable points at synchronous height correspond to the stable Lagrange points L_4 and L_5; the configuration is no longer an equilateral triangle since the two 'large' masses are now carried by the Earth instead of gravitating around one another. Since artificial satellites can be captured, this must also be so for naturally occurring dust particles. Moreover, it seems to be possible (Allan 1967) to show that there is a mechanism whereby particles become captured and eventually come to rest at the stable longitude. At synchronous height the orbital plane would be seriously perturbed both by the Earth's oblateness and by lunisolar effects; although this aspect has not been fully analysed it seems likely that the orbital planes of any captured particles would eventually tend to the stable plane which is inclined at 7·4° to the equator. It therefore seems a worthwhile experiment to see if any clouds of dust particles can be detected near the stable longitudes and lying in the stable plane for synchronous height.

5. Conclusions

The available observations of Syncom 2 and Syncom 3 have been analysed in an attempt to determine the longitude dependence of the geopotential, at least as it affects synchronous satellites. A total of 73 observations divided into seven arcs has been fitted by numerically integrated solutions of the equation of motion. Although the drift accelerations are very well determined near the longitudes 180° E and 300° E where the observations are clustered, the distribution in longitude is not good enough to give a satisfactory determination of the individual coefficients. Thus the solutions which give the lowest sum of squares of the residuals contain $J_{2,2}$ and $J_{3,1}$, and the values found for $J_{3,1}$ are high, since it must have little influence on the motion. Probably the solution containing $J_{2,2}$, $J_{3,3}$ and $J_{4,4}$ is more realistic, and this in fact gives values for $J_{2,2}$ and $\phi_{2,2}$ in good agreement with the four most recent determinations from close satellites. To improve the distribution in longitude it would be particularly valuable to have observations in the region 0 to 40° E and/or near either of the two stable points near 70° E and 250° E.

If there is or has been any significant population of dust particles in distant geocentric orbits, it is likely that a proportion will have been captured in the synchronous resonance, and will have accumulated near the stable positions. The size of any such particles would be limited by the periodic variation in eccentricity produced by radiation pressure; for black-body absorbers of density 3 g/cm³, the eccentricity could remain less than 0·1 if the radius of the particles were larger than about 3×10^{-3} cm. It is therefore suggested that it would be worth attempting to observe whether there are clouds of dust particles near the stable longitudes, and lying in the stable plane for synchronous height.

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